

## KITES AND PSEUDO BL-ALGEBRAS

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ABSTRACT. We investigate a construction of a pseudo BL-algebra out of an  $\ell$ -group called a kite. We show that many well-known examples of algebras related to fuzzy logics can be obtained in that way. We describe subdirectly irreducible kites. As another application, we exhibit a new countably infinite family of varieties of pseudo BL-algebras covering the variety of Boolean algebras.

## 1. INTRODUCTION

Lattice-ordered groups ( $\ell$ -groups) are intimately connected with certain algebras related to fuzzy logics. Indeed, the discovery of one such connection predates fuzzy logics, as it was made by Chang [Cha] in his algebraic proof of completeness of infinitely-valued Łukasiewicz's logic. Much later Mundici [Mun] proved a categorical equivalence between the variety of MV-algebras and the class of Abelian  $\ell$ -groups with strong unit. This was extended by Dvurečenskij [Dvu1] to an equivalence between the variety of *pseudo MV-algebras* (a noncommutative generalization of MV-algebras) and the class of all  $\ell$ -groups with strong unit. For *BL-algebras*, which constitute an algebraic semantics of classical fuzzy logic, the  $\ell$ -group connection was investigated by Aglianò and Montagna [AgMo] who proved that all linearly ordered BL-algebras consist of building blocks that are either negative cones of  $\ell$ -groups, or negative intervals in unital  $\ell$ -groups. A similar result was proved by Dvurečenskij for representable (i.e., such that subdirectly irreducibles are linearly ordered) pseudo BL-algebras (cf. [Dvu2]). Jipsen and Montagna [JiMo] constructed a subdirectly irreducible pseudo BL-algebra, which is not linearly ordered, yet it is made out of an  $\ell$ -group in a rather special way, which may be visualized as resembling a kite: a two-dimensional head joined to a one-dimensional tail. In this paper we generalize their construction, and show that many well-known examples of the algebras mentioned above can be seen as particular cases of the generalized construction. As our generalization consists essentially in allowing more dimensions for the head and tail, we call our algebras *kites*.

The paper is organized as follows. Basic notions and notations are in Section 2. Section 3 defines kites, they are always pseudo BL-algebras. It gathers the main properties of kites. In particular, some kites are pseudo MV-algebras. Section 4 presents a list of important kites. Subdirectly irreducible kites and their classification are described in Section 5. Very important kites are finite-dimensional ones

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which are studied in Section 6, where it is shown that the variety of pseudo BL-algebras generated by all kites is generated by all finite-dimensional kites. Finally, Section 7 shows some applications of theory of kites, in particular, countably many covers generated by kites of the variety of Boolean algebras in the variety of pseudo BL-algebras are presented.

## 2. BASIC NOTIONS AND NOTATION

In terminology and notation we will follow [GJKO], which is also our standard reference for undefined notions and details. All classes (varieties) of algebras we deal with in the paper can be viewed as subclasses (subvarieties) of *FL-algebras*, that is, algebras  $\mathbf{A} = \langle A; \wedge, \vee, \backslash, /, \cdot, 0, 1 \rangle$  of type  $(2, 2, 2, 2, 2, 0, 0)$  such that

- $\langle A; \wedge, \vee \rangle$  is a lattice,
- $\langle A; \cdot, \backslash, /, 1 \rangle$  is a residuated monoid,
- $0$  is an element of  $A$ .

The operations  $\backslash$  and  $/$  are called, respectively, *left division* (or *right residuation*) and *right division* (or *left residuation*). Two unary operations: *left negation*  $\sim x = x \backslash 0$ , and *right negation*  $-x = 0/x$ , are commonly used, and will play a role in the paper. Negations bind stronger than multiplication, which binds stronger than divisions, which in turn bind stronger than the lattice connectives. The following identities will be of some importance later.

- (1)  $1 \geq x$ ,
- (2)  $0 \leq x$ ,
- (3)  $1 = 0$ ,
- (4)  $x(x \backslash (x \wedge y)) = x \wedge y = ((x \wedge y)/x)x$ ,
- (5)  $x(x \backslash y) = x \wedge y = (y/x)x$ ,
- (6)  $x \backslash y \vee y \backslash x = 1 = y/x \vee x/y$ ,
- (7)  $x/(y \backslash x) = x \vee y = (x/y) \backslash x$ ,
- (8)  $x/((x \vee y) \backslash x) = x \vee y = (x/(x \vee y)) \backslash x$ ,
- (9)  $\sim -x = x = -\sim x$ ,
- (10)  $\sim -x = -\sim x$ ,
- (11)  $1 = x(x \backslash 1)$ ,
- (12)  $xy = yx$ .

We write FL for the variety of all FL-algebras. In general, if X-algebras are defined and happen to form a variety, we will use sans-serif X for that variety. Thus, FL<sub>i</sub> is the variety of FL<sub>i</sub>-algebras: these are FL-algebras satisfying (1); they are also known as *integral*. FL-algebras satisfying (2) are called FL<sub>o</sub>-algebras, or *zero-bounded*. FL-algebras that are both integral and zero-bounded are known as FL<sub>w</sub>-algebras. FL-algebras satisfying (3) are *residuated lattices*; this trick allows us to make the constant 0 disappear; it is especially useful for viewing lattice-ordered groups ( $\ell$ -groups) as a variety of FL-algebras.

The identities (4) and (5) are both known as *divisibility*; in varieties of integral FL-algebras (4) is equivalent to (5), but not so in general. FL-algebras satisfying divisibility in the form (4) are *GBL-algebras*; integral GBL-algebras are typically defined using (5) instead. This has the advantage of making lattice meet definable by means of (either of) the divisions. As integrality is derivable from (5), it is a natural and economical choice of an equational base. For more on GBL we refer the reader to [GaTs, JiMo]. The identities in (6) are together known as *prelinearity*.

$\text{FL}_w$ -algebras satisfying prelinearity and divisibility are *pseudo BL-algebras* (cf. e.g., [DGI1, DGI2]). This variety is of prime importance in this paper, since the kites from the title are certain pseudo BL-algebras constructed out of  $\ell$ -groups in a special way. We use **psBL** as a name for the variety of pseudo BL-algebras.

An important subvariety of **psBL** is the variety **psMV** of *pseudo MV-algebras*: these are pseudo BL-algebras satisfying (7), or, equivalently<sup>1</sup>, pseudo BL-algebras satisfying (9). Pseudo MV-algebras are categorically equivalent to the class of  $\ell$ -groups with a strong unit, as shown in [Dvu1]. The role of (8) with respect to (7) is analogous to that of (4) with respect to (5), namely, (8) it is a non-integral version of (7). In particular, (8) holds in  $\ell$ -groups, while (7) does not. For this to make sense, we need to interpret  $\ell$ -groups as residuated lattices, and this is done with the help of (11). Namely, the subvariety of residuated lattices satisfying (11) is term equivalent to the variety of  $\ell$ -groups, upon defining  $x^{-1} = x \backslash 1$  one way, and  $x \backslash y = x^{-1}y$ ,  $x/y = xy^{-1}$  the other.

Pseudo BL-algebras satisfying (10), a natural weakening of (9), are known by an unassuming name of *good*. For example, every pseudo MV-algebra is good. It was an open question for some time whether every pseudo BL-algebra was good (cf. [DGI2, Problem 3.21]). It was resolved in the negative in [DGK], and the counterexample was in fact a special type of what we now call a kite, defined in [JiMo].

Finally, commutative pseudo BL-algebras are just *BL-algebras* and commutative pseudo MV-algebras are *MV-algebras*. This reflects the order of discovery: “pseudo” varieties<sup>2</sup> were discovered as noncommutative generalizations of BL- and MV-algebras, respectively.

Let  $\mathbf{A}$  be an FL-algebra. Given  $a \in A$ , we define  $a^1 := a$  and  $a^{n+1} := a^n a$ , for  $n \geq 1$ . An element  $a \in A$  is said to be (i) *idempotent* if  $a^2 = a$ , (ii) *Boolean* if it is idempotent and  $\sim a = a = \sim \sim a$ . Let  $B(\mathbf{A})$  be the set of Boolean elements of  $\mathbf{A}$ . It is the greatest Boolean subalgebra of  $\mathbf{A}$ . If  $\mathbf{A}$  is a BL-algebra, by [DGI2, Prop 2.10], an element  $a \in A$  is Boolean iff  $a \vee \sim a = 1$  iff  $a \vee \sim a$ . Then  $\sim a = \sim a$ .

We now recall some basic facts about arithmetical and structural properties of FL-algebras, residuated lattices and pseudo BL-algebras that we wish to use freely later.

**Lemma 2.1.** *Let  $\mathbf{A}$  be an FL-algebra. Then the equivalences*

$$x \leq z/y \quad \text{iff} \quad xy \leq z \quad \text{iff} \quad y \leq x \backslash z$$

*hold for any  $x, y, z \in A$ . If  $\mathbf{A}$  is integral, then  $xy \leq x \wedge y$  always holds, so in particular  $x^2 \leq x$ ; moreover  $x \backslash y = 1 = y/x$  hold iff  $x \leq y$ . If  $\mathbf{A}$  is divisible and  $x \geq y$ , there exist elements  $z_1$  and  $z_2$  such that  $xz_1 = y = z_2x$ ; in fact  $z_1 = x \backslash y$  and  $z_2 = y/x$ . If  $\mathbf{A}$  is divisible and  $z$  is idempotent, then  $xz = x \wedge z = zx$  holds for any  $x \in A$ .*

Let  $\mathbf{A}$  be an FL-algebra. A *left conjugate* of an element  $x \in A$  by an element  $y \in A$  is the element  $y \backslash xy \wedge 1$ , and its *right conjugate* is  $yx/y \wedge 1$ . An element  $x \in A$  is *central* if  $yx = xy$  holds for all  $y \in A$ . By the last statement of Lemma 2.1 all idempotent elements in pseudo BL-algebras are central. A *filter* of  $\mathbf{A}$  is a set  $F \subseteq A$  such that (i)  $F$  is a subalgebra of 0-free reduct of  $\mathbf{A}$  and (ii)  $F$  is convex as an ordered set. A filter  $F$  is normal if it is closed under conjugates, i.e., for all

<sup>1</sup>Over pseudo BL-algebras; the equivalence does not hold in general.

<sup>2</sup>Not pseudovarieties, which are classes closed under finite direct products, subalgebras and homomorphic images.

$x \in F$  and all  $y \in A$  both  $y \setminus xy$  and  $yx/y$  belong to  $F$ . Normal filters are also called *convex normal subalgebras* (e.g. in [GJKO]), which makes good sense for residuated lattices but can be confusing for FL-algebras: convex normal subalgebras may not be subalgebras in the proper sense because they do not need to contain 0. If  $\mathbf{A}$  is integral, filters can be alternatively defined as subsets of  $A$  that are upward closed and closed under multiplication. For a set  $S \subseteq A$  we denote its upward closure by  $\uparrow S$ . If  $S = \{s\}$  we write  $\uparrow s$  instead of  $\uparrow\{s\}$ .

**Lemma 2.2.** *Let  $\mathbf{A}$  be an FL-algebra. For each congruence  $\theta$  on  $\mathbf{A}$ , its class  $1/\theta$  is a normal filter. Conversely, each normal filter  $F$  corresponds to a unique congruence  $\theta_F$  on  $\mathbf{A}$ . This correspondence establishes a lattice isomorphism between the lattices of normal filters of  $\mathbf{A}$  and congruences of  $\mathbf{A}$ . If  $\mathbf{A}$  is integral and  $z \in A$  is idempotent and central, then  $\uparrow z$  is a normal filter. If  $\mathbf{A}$  is, moreover, divisible and  $y \in A$  is idempotent, then  $\uparrow y$  is normal.*

### 3. KITES

Let  $\mathbf{G}$  be an  $\ell$ -group, and  $I, J$  be sets with  $|J| \leq |I|$ . Since only the cardinalities of  $I$  and  $J$  matter for the construction, it is harmless to think of these sets as ordinals. We will do so explicitly in the next section. Let further  $\lambda, \rho: J \rightarrow I$  be injections. Now we define an algebra with the universe  $(G^+)^J \uplus (G^-)^I$ . We order its universe by keeping the original coordinatewise ordering within  $(G^+)^J$  and  $(G^-)^I$ , and setting  $x \leq y$  for all  $x \in (G^+)^J, y \in (G^-)^I$ . It is easy to verify that this is a (bounded) lattice ordering of  $(G^+)^J \uplus (G^-)^I$ . Notice also that the case  $I = J$  is not excluded, so the element  $e^I$  may appear twice: at the bottom of  $(G^+)^J$  and at the top of  $(G^-)^I$ . To avoid confusion in the definitions below, we adopt a convention of writing  $a_i^{-1}, b_i^{-1}, \dots$  for elements of  $(G^-)^I$  and  $f_j, g_j, \dots$  for elements of  $(G^+)^J$ . In particular, we will write  $e^{-1}$  for  $e$  as an element of  $G^-$ . We also put 1 for the constant sequence  $(e^{-1})^I$  and 0 for the constant sequence  $e^J$ . With these conventions in place we are ready to define multiplication, putting:

$$\begin{aligned} \langle a_i^{-1}: i \in I \rangle \cdot \langle b_i^{-1}: i \in I \rangle &= \langle (b_i a_i)^{-1}: i \in I \rangle \\ \langle a_i^{-1}: i \in I \rangle \cdot \langle f_j: j \in J \rangle &= \langle a_{\lambda(j)}^{-1} f_j \vee e: j \in J \rangle \\ \langle f_j: j \in J \rangle \cdot \langle a_i^{-1}: i \in I \rangle &= \langle f_j a_{\rho(j)}^{-1} \vee e: j \in J \rangle \\ \langle f_j: j \in J \rangle \cdot \langle g_j: j \in J \rangle &= \langle e: j \in J \rangle = 0. \end{aligned}$$

**Definition 3.1.** Divisions,  $/$  and  $\backslash$ , corresponding to multiplication defined as above on  $(G^+)^J \uplus (G^-)^I$  are defined by:

$$\begin{aligned}
\langle a_i^{-1} : i \in I \rangle \backslash \langle b_i^{-1} : i \in I \rangle &= \langle a_i b_i^{-1} \wedge e^{-1} : i \in I \rangle \\
\langle b_i^{-1} : i \in I \rangle / \langle a_i^{-1} : i \in I \rangle &= \langle b_i^{-1} a_i \wedge e^{-1} : i \in I \rangle \\
\langle a_i^{-1} : i \in I \rangle \backslash \langle f_j : j \in J \rangle &= \langle a_{\lambda(j)} f_j : j \in J \rangle \\
\langle f_j : j \in J \rangle / \langle a_i^{-1} : i \in I \rangle &= \langle f_j a_{\rho(j)} : j \in J \rangle \\
\langle f_j : j \in J \rangle \backslash \langle g_j : j \in J \rangle &= \langle a_i^{-1} : i \in I \rangle, \\
\text{where } a_i^{-1} &= \begin{cases} f_{\rho^{-1}(i)}^{-1} g_{\rho^{-1}(i)} \wedge e^{-1} & \text{if } \rho^{-1}(i) \text{ is defined} \\ e^{-1} & \text{otherwise} \end{cases} \\
\langle g_j : j \in J \rangle / \langle f_j : j \in J \rangle &= \langle b_i^{-1} : i \in I \rangle, \\
\text{where } b_i^{-1} &= \begin{cases} g_{\lambda^{-1}(i)} f_{\lambda^{-1}(i)}^{-1} \wedge e^{-1} & \text{if } \lambda^{-1}(i) \text{ is defined} \\ e^{-1} & \text{otherwise,} \end{cases} \\
\langle a_i^{-1} : i \in I \rangle / \langle f_j : j \in J \rangle &= (e^{-1})^I = \langle f_j : j \in J \rangle \backslash \langle a_i^{-1} : i \in I \rangle.
\end{aligned}$$

We will call the algebra we have just defined a *kite* of  $\mathbf{G}$ , and write  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  for it. Observe that if we take  $I = J$ , then  $\lambda$  and  $\rho$  become permutations of the set of coordinates and so the kite construction is reminiscent of wreath product. This analogy is not mistaken, as we will see later. For the moment, let us focus on the algebra  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ . The next lemma shows that the algebra  $K_{I,J}^{\lambda,\rho}(\mathbf{G}) = \langle (G^+)^J \uplus (G^-)^I; \vee, \wedge, \cdot, \backslash, /, 1, 0 \rangle$  is a pseudo BL-algebra.

**Lemma 3.2.** For any  $\ell$ -group  $\mathbf{G}$  and any choice of appropriate sets  $I, J$  and maps  $\lambda, \rho$ , the algebra  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is a pseudo BL-algebra.

*Proof.* It is clear that  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is a lattice-ordered groupoid with unit. To show that multiplication is associative, first observe that triples from  $(G^-)^I$  associate because  $(G^-)^I$  is just the negative cone of  $\mathbf{G}^I$ . Next, triples involving at least two elements from  $(G^+)^J$  associate because both the products equal 0. The remaining cases all involve one element from  $(G^-)^I$  and two from  $(G^+)^J$ . One such case is:

$$\begin{aligned}
(\langle a_i^{-1} : i \in I \rangle \cdot \langle f_j : j \in J \rangle) \cdot \langle b_i^{-1} : i \in I \rangle &= \langle a_{\lambda(j)}^{-1} f_j \vee e : j \in J \rangle \cdot \langle b_i^{-1} : i \in I \rangle \\
&= \langle (a_{\lambda(j)}^{-1} f_j \vee e) b_{\rho(j)}^{-1} \vee e : j \in J \rangle \\
&= \langle a_{\lambda(j)}^{-1} f_j b_{\rho(j)}^{-1} \vee b_{\rho(j)}^{-1} \vee e : j \in J \rangle \\
&= \langle a_{\lambda(j)}^{-1} f_j b_{\rho(j)}^{-1} \vee e : j \in J \rangle \\
&= \langle a_{\lambda(j)}^{-1} f_j b_{\rho(j)}^{-1} \vee a_{\lambda(j)}^{-1} \vee e : j \in J \rangle \\
&= \langle a_{\lambda(j)}^{-1} (f_j b_{\rho(j)}^{-1} \vee e) : j \in J \rangle \\
&= \langle a_i : i \in I \rangle \cdot \langle f_j b_{\rho(j)}^{-1} \vee e : j \in J \rangle \\
&= \langle a_i^{-1} : i \in I \rangle \cdot (\langle f_j : j \in J \rangle \cdot \langle b_i^{-1} : i \in I \rangle).
\end{aligned}$$

Other cases follow by similar calculations. Now, to show that divisibility holds, we also proceed case by case. Let us deal with two cases here. The first is:

$$\begin{aligned} \langle f_j : j \in J \rangle \cdot (\langle f_j : j \in J \rangle \setminus \langle g_j : j \in J \rangle) &= \langle f_j : j \in J \rangle \cdot \langle a_i^{-1} : i \in I \rangle \\ &= \langle f_j a_{\rho(j)}^{-1} \vee e : j \in J \rangle \end{aligned}$$

where

$$a_i^{-1} = \begin{cases} f_{\rho^{-1}(i)}^{-1} g_{\rho^{-1}(i)} \wedge e^{-1} & \text{if } \rho^{-1}(i) \text{ is defined} \\ e^{-1} & \text{otherwise} \end{cases}$$

but, observe that  $\rho^{-1}(\rho(j))$  is always defined and equals  $j$ , so calculating further we obtain  $f_j a_{\rho(j)}^{-1} = g_j \wedge f_j$  for every  $j \in J$ , and therefore

$$\langle f_j a_{\rho(j)}^{-1} \vee e : j \in J \rangle = \langle (f_j \wedge g_j) \vee e : j \in J \rangle = \langle f_j \wedge g_j : j \in J \rangle$$

as required. For the second, take:

$$\begin{aligned} \langle a_i^{-1} : i \in I \rangle \cdot (\langle a_i^{-1} : i \in I \rangle \setminus \langle f_j : j \in J \rangle) &= \langle a_i^{-1} : i \in I \rangle \cdot \langle a_{\lambda(j)} f_j : j \in J \rangle \\ &= \langle a_{\lambda(j)}^{-1} a_{\lambda(j)} f_j \vee e : j \in J \rangle \\ &= \langle f_j : j \in J \rangle. \end{aligned}$$

All other cases are straightforward. It remains to show prelinearity. Since multiplication and divisions in  $(G^-)^I$  are defined coordinatewise, prelinearity for  $x, y \in (G^-)^I$  is inherited from  $G^-$ . If  $x \in (G^-)^I$  and  $y \in (G^+)^J$ , or *vice versa* prelinearity holds trivially. For the only remaining case, calculating

$$\langle f_j : j \in J \rangle \setminus \langle g_j : j \in J \rangle \vee \langle g_j : j \in J \rangle \setminus \langle f_j : j \in J \rangle$$

yields two cases: (1) if  $\rho^{-1}(i)$  is defined, we have

$$\begin{aligned} &f_{\rho^{-1}(i)}^{-1} g_{\rho^{-1}(i)} \wedge e^{-1} \vee g_{\rho^{-1}(i)}^{-1} f_{\rho^{-1}(i)} \wedge e^{-1} \\ &= (f_{\rho^{-1}(i)}^{-1} g_{\rho^{-1}(i)} \vee g_{\rho^{-1}(i)}^{-1} f_{\rho^{-1}(i)}) \wedge e^{-1} \\ &= e^{-1} \end{aligned}$$

and (2) if  $\rho^{-1}(i)$  is not defined, we have

$$a_i^{-1} \vee a_i^{-1} = e^{-1} \vee e^{-1} = e^{-1}$$

as well. Thus, prelinearity holds and that finishes the proof of all the claims in the lemma.  $\square$

Somewhat surprisingly, many kites turn out to be pseudo MV-algebras.

**Lemma 3.3.** *Let  $\mathbf{G}$  be an  $\ell$ -group, and suppose  $|I| = |J|$  and  $\lambda, \rho$  are bijections. Then  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is a pseudo MV-algebra.*

*Proof.* By Lemma 3.2 we only need to show that under the conditions of the lemma, the identity  $x/(y \setminus x) = x \vee y = (x/y) \setminus x$  holds. We have two nontrivial cases to consider:

*Case 1.*  $x \in (G^+)^J$  and  $y \in (G^-)^I$ . Then,  $x = \langle f_j : j \in J \rangle \leq \langle a_i^{-1} : i \in I \rangle = y$ , and so we calculate:

$$\begin{aligned} \langle f_j : j \in J \rangle / (\langle a_i^{-1} : i \in I \rangle \setminus \langle f_j : j \in J \rangle) &= \langle f_j : j \in J \rangle / \langle a_{\lambda(j)} f_j : j \in J \rangle \\ &= \langle f_{\lambda^{-1}(i)} (a_{\lambda(j)} f_j)_{\lambda^{-1}(i)}^{-1} \wedge e^{-1} : i \in I \rangle \\ &= \langle f_{\lambda^{-1}(i)} f_{\lambda^{-1}(i)}^{-1} a_i^{-1} : i \in I \rangle \\ &= \langle a_i^{-1} : i \in I \rangle \end{aligned}$$

which shows that  $x/(y \setminus x) = y = x \vee y$  holds. Notice that we used bijectiveness of  $\lambda$  to pass from the first to the second equality above.

*Case 2.*  $x, y \in (G^+)^J$ . Then,  $x = \langle f_j : j \in J \rangle$  and  $y = \langle g_j : j \in J \rangle$ , and so we calculate:

$$\begin{aligned} \langle f_j : j \in J \rangle / (\langle g_j : j \in J \rangle \setminus \langle f_j : j \in J \rangle) &= \langle f_j : j \in J \rangle / \langle g_{\rho^{-1}(i)} f_{\rho^{-1}(i)} \wedge e^{-1} : i \in I \rangle \\ &= \langle f_j (g_{\rho^{-1}(i)} f_{\rho^{-1}(i)} \wedge e^{-1})_{\rho(j)}^{-1} : j \in J \rangle \\ &= \langle f_j (f_j^{-1} g_j \vee e) : j \in J \rangle \\ &= \langle g_j \vee f_j : j \in J \rangle \end{aligned}$$

which again shows that  $x/(y \setminus x) = x \vee y$  holds. The proofs for  $x \vee y = (x/y) \setminus x$  are symmetric.  $\square$

As we already mentioned, it was an open problem ([DGI2, Problem 3.21]) for a while whether every pseudo BL-algebra is good. We found a negative solution in [DGK] using special types of kites from [JiMo]. Below we characterize good kites, and in the last section we will exhibit a countably infinite family of varieties of kites with the property that their all and only good members are Boolean algebras.

**Lemma 3.4.** *Let  $\mathbf{G}$  be an  $\ell$ -group, and  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  a kite.*

- (1)  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is good if and only if  $\lambda(J) = \rho(J)$ .
- (2)  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is a pseudo MV-algebra if and only if  $\lambda(J) = I = \rho(J)$ .

*Proof.* (1) Let  $x = \langle a_i : i \in I \rangle$ . Then  $\neg x = \langle a_{\rho(j)} : j \in J \rangle$  and  $\sim x = \langle a_{\lambda(j)} : j \in J \rangle$ . In addition,  $\sim \neg x = \langle z_i : i \in I \rangle$ , where

$$z_i^{-1} = \begin{cases} a_i^{-1} & \text{if } \lambda^{-1}(i) \text{ is defined} \\ e^{-1} & \text{otherwise,} \end{cases}$$

and  $\neg \sim x = \langle y_i : i \in I \rangle$ , where

$$y_i^{-1} = \begin{cases} a_i^{-1} & \text{if } \rho^{-1}(i) \text{ is defined} \\ e^{-1} & \text{otherwise.} \end{cases}$$

Now, if  $x = \langle f_j : j \in J \rangle$ , we have  $\sim \neg x = \langle g_j : j \in J \rangle$ , where

$$g_j = \begin{cases} f_j & \text{if } \rho^{-1}(i) \text{ is defined} \\ e & \text{otherwise,} \end{cases}$$

and  $\neg \sim x = \langle h_j : j \in J \rangle$ , where

$$h_j = \begin{cases} f_j & \text{if } \lambda^{-1}(i) \text{ is defined} \\ e & \text{otherwise.} \end{cases}$$

Hence, if  $\lambda(J) = \rho(J)$ , the kite  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is good.

Conversely, assume that the kite  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is good, and let  $\lambda(J) \neq \rho(J)$ . For our aims we can assume that each  $a_i^{-1} \neq e^{-1}$ . There is an  $i \in I$  such that either  $\lambda^{-1}(i)$  or  $\rho^{-1}(i)$  is not defined. Equivalently,  $x_i^{-1} = e^{-1}$  and  $x_i^{-1} = a_i^{-1}$  or  $y_i^{-1} = e^{-1}$  and  $y_i^{-1} = a_i^{-1}$ . Hence,  $\lambda(J) = \rho(J)$ .

(2) If  $\lambda(J) = I = \rho(J)$ , then  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is a pseudo MV-algebra by Lemma 3.3. Conversely, let the kite be a pseudo MV-algebra. Since every pseudo MV-algebra is good, by the first part of the present proof, we have  $\lambda(J) = \rho(J)$ . Now assume by absurd that there is an  $i \in I \setminus \lambda(J)$ . Then both  $\lambda^{-1}(i)$  and  $\rho^{-1}(i)$  are not defined and whence  $x_i^{-1} = e^{-1} = y_i^{-1} \neq a_i$  which contradicts the property  $\sim x = x = \sim \sim x$ . Whence  $\lambda(J) = I = \rho(J)$ .  $\square$

**3.1. One-dimensional elements.** One property of kites that is frequently used in calculations is that double negations amount to certain shifts (often, rotations) of coordinate system. Below, we will state this observation in the form that will later help characterize finite-dimensional kites.

Let  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  be a kite. An element  $a \in (G^-)^I$  will be called  $\alpha$ -dimensional, for some cardinal  $\alpha$ , if  $|\{i \in I : a(i) \neq e\}| = \alpha$ . Similarly, this notion applies to elements of  $(G^+)^J$ . One-dimensional elements are particularly easy to work with, and, moreover, it is immediately seen that every element of a kite is a join or a meet of one-dimensional elements.

**Lemma 3.5.** *Let  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  be a kite and  $a \in (G^-)^I$  be one-dimensional, with  $a(i) \neq e^{-1}$ . Then  $\sim \sim a$ ,  $\sim -a$ ,  $-\sim a$ , and  $--a$  are at most one-dimensional. If  $\lambda^{-1}(i)$  and  $\rho^{-1}(i)$  are defined, these elements are exactly one-dimensional. More precisely, the following hold:*

- (1)  $\sim \sim a < 1$  and  $-\sim a < 1$  iff  $\lambda^{-1}(i)$  is defined,
- (2)  $--a < 1$  and  $\sim -a < 1$  iff  $\rho^{-1}(i)$  is defined,
- (3)  $-\sim a = a$  iff  $\lambda^{-1}(i)$  is defined,
- (4)  $\sim -a = a$  iff  $\rho^{-1}(i)$  is defined,
- (5)  $\sim \sim a \vee a = 1$  iff  $\rho(\lambda^{-1}(i)) \neq i$ ,
- (6)  $--a \vee a = 1$  iff  $\lambda(\rho^{-1}(i)) \neq i$ .

*Proof.* To begin with, since  $\sim -a \geq a$  and  $-\sim a \geq a$  hold in any residuated lattice,  $\sim -a$  and  $-\sim a$  are at most one-dimensional. Let us calculate  $\sim \sim a$ . To make the notation less cumbersome, we first regard  $a$  as a sequence  $\langle e^{-1}, \dots, a^{-1}(i), e^{-1}, \dots \rangle$ , and then write  $\langle a^{-1}(i) \rangle$  for  $\langle e^{-1}, \dots, a^{-1}(i), e^{-1}, \dots \rangle$  and  $\langle a(i) \rangle$  for  $\langle e, \dots, a(i), e, \dots \rangle$ . Similarly, we put  $\langle e \rangle$  for  $e^J = 0$  and  $\langle e^{-1} \rangle$  for  $(e^{-1})^I = 1$ . Calculating  $\langle a^{-1}(i) \rangle \setminus \langle e \rangle$  we see that it is different from  $e$  at coordinate  $\lambda(j)$  if and only if  $\lambda(j) = i$ . Therefore

$$\langle a^{-1}(i) \rangle \setminus \langle e \rangle = \begin{cases} \langle a(\lambda^{-1}(i)) \rangle & \text{if } \lambda^{-1}(i) \text{ is defined} \\ \langle e \rangle & \text{otherwise.} \end{cases}$$

Then, calculating  $\langle a(\lambda^{-1}(i)) \rangle \setminus \langle e \rangle$  in turn, we get that it is different from  $e^{-1}$  at coordinate  $k$  if and only if  $\rho^{-1}(k)$  is defined and equal to  $\lambda^{-1}(i)$ , that is if and only if  $k = \rho(\lambda^{-1}(i))$ . Altogether, we have

$$(\langle a^{-1}(i) \rangle \setminus \langle e \rangle) \setminus \langle e \rangle = \begin{cases} \langle a^{-1}(\rho \circ \lambda^{-1}(i)) \rangle & \text{if } \lambda^{-1}(i) \text{ is defined} \\ \langle e^{-1} \rangle & \text{otherwise.} \end{cases}$$



By similar calculations we also obtain

$$\begin{aligned} \langle e \rangle / (\langle a^{-1}(i) \rangle \setminus \langle e \rangle) &= \begin{cases} \langle a^{-1}(i) \rangle & \text{if } \lambda^{-1}(i) \text{ is defined} \\ \langle e^{-1} \rangle & \text{otherwise} \end{cases} \\ (\langle e \rangle / \langle a^{-1}(i) \rangle) \setminus \langle e \rangle &= \begin{cases} \langle a^{-1}(i) \rangle & \text{if } \rho^{-1}(i) \text{ is defined} \\ \langle e^{-1} \rangle & \text{otherwise} \end{cases} \\ \langle e \rangle / (\langle e \rangle / \langle a^{-1}(i) \rangle) &= \begin{cases} \langle a^{-1}(\lambda \circ \rho^{-1}(i)) \rangle & \text{if } \rho^{-1}(i) \text{ is defined} \\ \langle e^{-1} \rangle & \text{otherwise.} \end{cases} \end{aligned}$$

Then all the claims follow from these calculations and coordinatewise ordering of  $(G^-)^I$ . To give just one example, for (5) we have  $\sim \sim a \vee a = 1$  if and only if  $\langle a^{-1}(\rho \circ \lambda^{-1}(i)) \rangle \vee \langle a^{-1}(i) \rangle = \langle e^{-1} \rangle$  if and only if  $\rho \circ \lambda^{-1}(i) \neq i$ .  $\square$

#### 4. EXAMPLES OF KITES

As we have already remarked, in the kite construction we can think of the index sets  $I$  and  $J$  as ordinals. Doing so systematically also makes classification of kites easier, so throughout this section we assume that  $I$  and  $J$  are ordinals. Below we give five examples of rather familiar algebras that can be rendered as kites.

**4.1. Boolean algebras.** Let  $I = 0 = J$ . Then,  $(G^-)^I$  and  $(G^+)^J$  are both singletons, and  $\lambda = \rho$  can only be the empty function (hence, an injection). Thus,  $K_{0,0}^{\emptyset,\emptyset}(\mathbf{G})$  is the two-element Boolean algebra for any  $\ell$ -group  $\mathbf{G}$ . We also get the two-element Boolean algebra in another way. Namely, let  $\mathbf{O}$  be the trivial  $\ell$ -group. Then  $\lambda = id = \rho$ , and  $K_{I,J}^{id,id}(\mathbf{O})$  is the two-element Boolean algebra for any choice of  $I$  and  $J$ .

**4.2. Product logic algebras.** Let  $I = 1$  and  $J = 0$ . As the only function from  $J$  to  $I$  is the empty function (which is an injection), the kite  $K_{1,0}^{\emptyset,\emptyset}(\mathbf{G})$  is well-defined for any  $\ell$ -group  $\mathbf{G}$ . Let  $\mathbb{R}^+$  stand for the  $\ell$ -group of positive reals under multiplication. Then  $K_{1,0}^{\emptyset,\emptyset}(\mathbb{R}^+)$  is isomorphic to the *standard product logic algebra*, i.e., the real interval  $[0, 1]$  with usual multiplication, and divisions given by  $x/y = \frac{x}{y} = y/x$  for  $y \neq 0$  and  $x/0 = 1 = 0/x$ . Taking  $\mathbb{Z}$  for  $\mathbf{G}$ , we obtain as  $K_{1,0}^{\emptyset,\emptyset}(\mathbb{Z})$  the algebra  $\mathbb{Z}_\perp^- = \langle \mathbb{Z}^- \cup \{\perp\}; \max, \min, +, -, \perp, 0 \rangle$ , where  $\perp = -\infty$ . This is also a product logic algebra. Both  $K_{1,0}^{\emptyset,\emptyset}(\mathbb{R}^+)$  and  $K_{1,0}^{\emptyset,\emptyset}(\mathbb{Z})$  generate the whole variety of product logic algebras. This variety covers the variety of Boolean algebras.

**4.3. Jipsen-Montagna algebras.** Taking  $I = 2$ ,  $J = 1$ ,  $\lambda(0) = 0$ , and  $\rho(0) = 1$ , we get that  $K_{2,1}^{\lambda,\rho}(\mathbf{G})$  is a Jipsen-Montagna algebra, for any  $\ell$ -group  $\mathbf{G}$ . In particular, the algebra  $K_{2,1}^{\lambda,\rho}(\mathbb{Z})$  is a pseudo BL-algebra which is not good [DGK] and it generates another cover of the variety of Boolean algebras. In Section 7 we will show that the same holds for  $K_{n+1,n}^{\lambda,\rho}(\mathbb{Z})$  with  $\lambda(i) = i$  and  $\rho(i) = i + 1$ , for an arbitrary  $n \in \omega$ .

**4.4. Chang chain and its subdirect powers.** Now, let  $I = 1 = J$ . Then  $K_{1,1}^{id,id}(\mathbb{Z})$  is the Chang chain, denoted  $\mathbf{S}_1^\omega$  in Komori's paper, and  $\mathbf{C}_\infty$  in [GJKO]. The variety generated by  $K_{1,1}^{id,id}(\mathbb{Z})$  is also a cover of the variety of Boolean algebras.

For  $I = J > 1$ , the kite  $K_{I,I}^{id,id}(\mathbb{Z})$  is subdirectly embeddable in  $(\mathbf{C}_\infty)^I$  and thus  $\mathbf{V}(K_{1,1}^{id,id}(\mathbb{Z})) = \mathbf{V}(K_{I,I}^{id,id}(\mathbb{Z}))$ .

**4.5. Intervals in Scrimger groups.** Taking  $I = J = n$  for  $n \geq 2$ , and putting  $\lambda(i) = i$  and  $\rho(i) = i+1 \pmod{n}$ , we get that  $K_{I,I}^{\lambda,\rho}(\mathbb{Z})$  is isomorphic to  $\Gamma(\mathbf{G}_n, (\langle 0 \rangle, 1))$ , where  $\mathbf{G}_n$  is the subgroup of  $\mathbb{Z} \wr \mathbb{Z}$  (antilexicographically ordered), consisting of the elements  $\langle \langle a_i : i \in \mathbb{Z} \rangle, b \rangle$ , such that  $i = j \pmod{n}$  implies  $a_i = a_j$ .

## 5. SUBDIRECTLY IRREDUCIBLE KITES

We will characterise subdirectly irreducible kites and show they fall into five broad classes. To begin with, we state a few facts on normal filters in kites.

**Lemma 5.1.** *Let  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  be a kite. Then  $(G^-)^I$  is a maximal normal filter of  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ .*

*Proof.* It is clear that  $(G^-)^I$  is a maximal filter, so we only need to show that  $(G^-)^I$  is closed under conjugation by elements outside  $(G^-)^I$ . Let  $x \in (G^-)^I$  and  $y \in (G^+)^J$ . Taking  $y \backslash xy$  we observe that  $xy \in (G^+)^J$ . By definition of divisions, we then get that  $y \backslash xy \in (G^-)^I$ . By symmetry the same holds for right conjugates.  $\square$

**Lemma 5.2.** *Let  $\mathbf{G}$  be an  $\ell$ -group, and  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  a kite. Let  $F$  be a convex normal subgroup of  $\mathbf{G}$ , and  $N = F|_{G^-}$ . Then  $N^I = \{\langle a_i^{-1} : i \in I \rangle : a_i^{-1} \in N\}$  is a normal filter of  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ .*

*Proof.* Since multiplication and order are defined coordinatewise on  $(G^-)^I$ , it is obvious that  $N^I$  is a filter. We need to show that  $N^I$  is closed under conjugates. Now, conjugation by an element of  $(G^-)^I$  also proceeds coordinatewise, so  $N^I$  is closed under all such conjugates. Take an element  $\langle f_j : j \in J \rangle$  and consider  $\langle f_j : j \in J \rangle \backslash \langle a_i^{-1} : i \in I \rangle \cdot \langle f_j : j \in J \rangle$ , for some  $\langle a_i^{-1} : i \in I \rangle \in N^I$ . This is equal to  $\langle f_j : j \in J \rangle \backslash \langle a_{\lambda(j)}^{-1} f_j \vee e : j \in J \rangle$ , which in turn equals  $\langle b_i^{-1} : i \in I \rangle$ , where

$$b_i^{-1} = \begin{cases} (f_{\rho^{-1}(i)}^{-1} a_{\lambda \circ \rho^{-1}(i)}^{-1} f_{\rho^{-1}(i)} \wedge e^{-1}) \vee f_{\rho^{-1}(i)}^{-1} & \text{if } \rho^{-1}(i) \text{ is defined} \\ e^{-1} & \text{otherwise} \end{cases} \quad (0)$$

as one can verify by a series of simple calculations. Notice that  $f_{\rho^{-1}(i)}^{-1} a_{\lambda \circ \rho^{-1}(i)}^{-1} f_{\rho^{-1}(i)}$  is a conjugate of a member of  $N$  and thus belongs to  $F$  by normality. Therefore  $f_{\rho^{-1}(i)}^{-1} a_{\lambda \circ \rho^{-1}(i)}^{-1} f_{\rho^{-1}(i)} \wedge e^{-1}$  belongs to  $N$ . Thus, by upward closedness of  $N$  we get that  $b_i \in N$ , for each  $i \in I$ , and so  $\langle b_i^{-1} : i \in I \rangle \in N^I$ . This shows that  $N^I$  is normal, as required.  $\square$

The converse of Lemma 5.2 is also true, but it will be useful to define a technical notion before we show it. Namely, for a proper normal filter  $N$  of a kite  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ , and a  $k \in I$ , we define the restriction of  $N$  to a coordinate  $k$  as the set of elements  $a \in N$  with  $a(i) \neq e^{-1}$  iff  $i = k$ .

**Lemma 5.3.** *Let  $\mathbf{G}$  be an  $\ell$ -group,  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  a kite, and  $N$  a normal filter of  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ . For any  $k \in I$ , the restriction of  $N$  to  $k$  is a normal filter of  $\mathbf{G}^-$ , and therefore the negative part of a convex normal subgroup of  $\mathbf{G}$ .*

*Proof.* Since  $N$  is proper, it is contained in  $(G^-)^I$ , which is a direct power of  $\mathbf{G}^-$ . Then, the restriction of  $N$  to  $k$  is just the  $k$ -th projection of  $N$ , and the claim follows.  $\square$

The next lemma shows that subdirectly irreducible kites can only arise out of subdirectly irreducible  $\ell$ -groups, and a first characterization of subdirectly irreducible kites follows.

**Lemma 5.4.** *Let  $\mathbf{G}$  be an  $\ell$ -group, and  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  a kite. If  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is subdirectly irreducible, so is  $\mathbf{G}$ .*

*Proof.* We will prove the contrapositive. Suppose  $\mathbf{G}$  is not subdirectly irreducible. Then, there exists a set  $\{\mathbf{M}_s : s \in S\}$  of nontrivial convex normal subgroups of  $\mathbf{G}$ , such that  $\bigcap_{s \in S} M_s = \{e\}$ . Let  $M_s^- = M_s|_{G^-}$ . Then, by Lemma 5.2,  $(M_s^-)^I$  is a normal filter of  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ , for each  $s \in S$ . Suppose  $\langle a_i : i \in I \rangle$  belongs to  $(M_s^-)^I$  for each  $s \in S$ . Then, for any coordinate  $k \in I$  we have that  $a_k \in M_s^-$  for all  $s \in S$ , and thus  $a_k = e$ . Therefore,  $\bigcap_{s \in S} (M_s^-)^I = \{e^I\}$ , showing that the set  $\{(M_s^-)^I : s \in S\}$  of nontrivial normal filters of  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  intersects trivially. Thus,  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is not subdirectly irreducible, proving the claim.  $\square$

**Theorem 5.5.** *Let  $\mathbf{G}$  be an  $\ell$ -group, and  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  a kite. The following are equivalent:*

- (1)  $\mathbf{G}$  is subdirectly irreducible and for all  $i, j \in I$  there exists  $m \in \omega$  such that  $(\rho \circ \lambda^{-1})^m(i) = j$  or  $(\lambda \circ \rho^{-1})^m(i) = j$ .
- (2)  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is subdirectly irreducible.

*Proof.* To prove that (1) implies (2), let  $\mathbf{M}$  be the smallest nontrivial convex normal subgroup of  $\mathbf{G}$ , and  $N$  be the restriction of  $M$  to the negative cone of  $\mathbf{G}$ . Then,  $N$  is the smallest nontrivial normal filter of  $\mathbf{G}^-$ . By Theorem 5.2 we have that  $N^I$  is a normal filter of  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ , so it remains to prove that  $N^I$  is the smallest nontrivial such. Since for any  $a \in N^I \setminus \{1\}$  there is a one-dimensional element  $a'$  with  $a \leq a' < 1$ , it suffices to prove that any one-dimensional element  $b \in N^I \setminus \{1\}$  generates  $N^I$ . Without loss of generality assume  $b = \langle b_0^{-1}, e^{-1}, \dots \rangle$ ; this is always achievable by a suitable re-ordering of  $I$ , regardless of its cardinality. Observe that  $b_0^{-1}$  generates  $N$ , since  $N$  is the smallest nontrivial normal filter of  $\mathbf{G}^-$ . It follows that  $b$  generates all members of  $N^I$  of the form  $\langle a^{-1}, e^{-1}, \dots \rangle$ , using only conjugates of the same form. Consider an arbitrary  $i \in I$ . By assumption, there is  $m \in \omega$  with  $\rho(\lambda^{-1})^m(0) = i$  or  $\lambda(\rho^{-1})^m(0) = i$ . Now, repeating  $m$  times the calculation from the proof of Lemma 3.5, we obtain that for an element  $u = \langle a^{-1}, e^{-1}, \dots \rangle$ , one of the following must be the case:

- if  $\rho(\lambda^{-1})^m(0) = i$ , then  $\underbrace{\sim \dots \sim}_{2m\text{-times}} u = \langle e^{-1}, \dots, e^{-1}, a^{-1}, e^{-1}, \dots \rangle$ ,
- if  $\lambda(\rho^{-1})^m(0) = i$ , then  $\underbrace{- \dots -}_{2m\text{-times}} u = \langle e^{-1}, \dots, e^{-1}, a^{-1}, e^{-1}, \dots \rangle$ ,

where  $a^{-1}$  occurs at a coordinate  $i$ ; again by a suitable renumbering of  $I$  we can assume it to be the  $m$ -th coordinate. By taking appropriate meets it then follows that every element of  $N^I$  can be generated, which proves the claim.

For the the converse, by Lemma 5.4 we can assume  $\mathbf{G}$  is subdirectly irreducible. Then, suppose there are  $i, j \in I$  such that for all  $m \in \omega$  we have  $\rho(\lambda^{-1})^m(i) \neq j$  and  $\lambda(\rho^{-1})^m(i) \neq j$ . We will call such  $i$  and  $j$  *disconnected*; otherwise,  $i$  and  $j$  will

be called *connected*. If all distinct members of a  $K \subseteq I$  are connected, we will call  $K$  a *connected component* of  $I$ . Now, let  $I_0$  and  $I_1$  be connected components of  $I$  such that  $i \in I_0$  and  $j \in I_1$ . Clearly,  $I_0$  and  $I_1$  are disconnected, that is no member of  $I_0$  is connected to any member of  $I_1$ . We will prove that  $N^{I_0} \cap N^{I_1} = \{1\}$ , from which it follows immediately that  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is not subdirectly irreducible. In fact, it suffices to show that for an element  $u = \langle a_i^{-1} : i \in I \rangle$  such that  $a_i^{-1} = e^{-1}$  for all  $i \notin I_0$ , and for any element  $b$ , the conjugate  $b \backslash ub$  has  $b \backslash ub(i) = e^{-1}$  if  $i \notin I_0$ , and the same holds for  $bu/b$ . Take  $b = \langle b_j : j \in J \rangle$ . We have

$$\begin{aligned} b \backslash ub &= \langle b_j : j \in J \rangle \backslash \langle a_i^{-1} : i \in I \rangle \cdot \langle b_j : j \in J \rangle \\ &= \langle b_j : j \in J \rangle \backslash \langle a_{\lambda(j)}^{-1} b_j \vee e : j \in J \rangle \\ &= \langle c_i^{-1} : i \in I \rangle \end{aligned}$$

$$\text{where } c_i^{-1} = \begin{cases} b_{\rho^{-1}(i)}^{-1} (a_{\lambda(\rho^{-1}(i))} b_{\rho^{-1}(i)} \vee e) \wedge e^{-1} & \text{if } \rho^{-1}(i) \text{ is defined} \\ e^{-1} & \text{otherwise.} \end{cases}$$

Now, by assumption  $a_i^{-1} = e^{-1}$  for  $i \notin I_0$ , and by connectedness,  $\lambda(\rho^{-1}(i)) \notin I_0$  if  $i \notin I_0$ . Therefore,  $c_i^{-1}$  can be different from  $e^{-1}$  only if  $i \in I_0$ , and thus  $b \backslash ub$  is of the required form. The claim for the other conjugate follows by symmetry.  $\square$

If  $I$  and  $J$  are finite, the form of subdirectly irreducible kites is even more restricted than Theorem 5.5 explicitly states. In such a case,  $I$  can only be the same size as  $J$  or bigger by one, and  $\lambda$  and  $\rho$  are essentially determined by the sizes of  $I$  and  $J$ .

**Lemma 5.6.** *If  $\mathbf{G}$  is an  $\ell$ -group,  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  a subdirectly irreducible kite, and  $I$  and  $J$  are finite, then  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is isomorphic to one of:*

- (1)  $K_{n,n}^{\lambda,\rho}(\mathbf{G})$ , with  $\lambda(j) = j$  and  $\rho(j) = j + 1 \pmod{n}$ ,
- (2)  $K_{n+1,n}^{\lambda,\rho}(\mathbf{G})$ , with  $\lambda(j) = j$  and  $\rho(j) = j + 1$ .

*Proof.* Assume first that  $|I| = n = |J|$ . Then, since  $\lambda$  and  $\rho$  are injections, we can number the elements of  $I$  and  $J$  so that  $\lambda(j) = j$ . If  $\rho(k) = k$  for some  $k$ , then  $\{k\}$  is a connected component of  $I$  and thus, by Theorem 5.5,  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is not subdirectly irreducible, contradicting the assumption. So,  $\rho(j) \neq j$ , for every  $j$ . We can then renumber  $I$  and  $J$  so that  $\rho(j) = j + 1 \pmod{n}$ .

For the second part, assume  $|J| < |I|$ . Then, there is  $i \in I$  not in the range of  $\rho$ . We start numbering  $I$  by putting  $i = 0$ . By Theorem 5.5 we get that  $I$  is connected, and so  $0 \in I$  is in the range of  $\lambda$ . We start numbering  $J$  by putting  $\lambda(0) = 0$ . Now,  $\rho(0) \neq 0$ , so we put  $\rho(0) = 1$ . Then, there are two cases to consider.

*Case 1.*  $\lambda^{-1}(1)$  is not defined. Then, if  $J$  contains a nonempty subset  $J'$  disconnected with  $\{0, 1\}$ , Theorem 5.5 yields a contradiction. Thus,  $J = \{0, 1\}$  and, since  $\lambda$  and  $\rho$  are injections,  $I = \{0\}$ . The claim holds in this case.

*Case 2.*  $\lambda^{-1}(1)$  is defined. Then, we extend the numbering of  $J$  putting  $\lambda^{-1}(1) = 1$ . Since  $|J| < |I|$ , there is  $j \in J \setminus \{0, 1\}$ . Now, if  $\rho(1) = 0$ , then  $\{0, 1\}$  and  $j$  are disconnected, contradicting subdirect irreducibility. Therefore,  $\rho(1) \neq 0$ , and by injectiveness  $\rho(1) \neq 1$ . We then put  $\rho(1) = 2$  extending the numbering of  $I$ .

Then, we repeat the procedure recursively, and by finiteness it must terminate. By inspection of the two cases, it is clear that it terminates in Case 1, for  $j \in J$  such that  $\lambda^{-1}(j)$  is not defined. Observe that  $j = n + 1 = \rho(n)$ , where  $n$  was numbered

at the immediately preceding stage employing Case 2. This results in the required numbering of  $I$  and  $J$  and at the same time shows that  $|I| = |J| + 1$ .  $\square$

Kites of the form  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  with  $I$  and  $J$  finite will be called *finite-dimensional*. We will deal with these in more detail in the next section. Next, we focus on subdirectly irreducible kites with  $I$  or  $J$  infinite.

**Lemma 5.7.** *Let  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  be a subdirectly irreducible kite. Then,  $I$  and  $J$  are at most countably infinite.*

*Proof.* Suppose  $I$  is uncountable, but  $J$  is countable. Then,  $\lambda(J)$  and  $\rho(J)$  are also countable, and therefore  $I \setminus (\lambda(J) \cup \rho(J))$  is nonempty. Thus for any  $k$  belonging to that set, we have that for  $i = k = j$  neither of the conditions stated in the second part of Theorem 5.5(2) can hold.

Suppose  $J$  is uncountable, and thus so is  $I$  because  $\lambda$  and  $\rho$  are injections. Take  $i \in \lambda(J)$ . The set  $\bigcup_{m \in \omega} (\rho \circ \lambda^{-1})^m(i)$  is also countable and therefore  $I \setminus \bigcup_{m \in \omega} (\rho \circ \lambda^{-1})^m(i)$  is uncountable. If  $i \notin \rho(J)$ , then any pair  $(i, j)$  with  $j \in I \setminus \bigcup_{m \in \omega} (\rho \circ \lambda^{-1})^m(i)$  witnesses a failure of the conditions from the second part of Theorem 5.5(2). If  $i \in \rho(J)$ , then, since  $\bigcup_{m \in \omega} (\lambda \circ \rho^{-1})^m(i)$  is countable as well, we have a  $k \in I \setminus \bigcup_{m \in \omega} (\lambda \circ \rho^{-1})^m(i)$ . Then, the pair  $(i, k)$  witnesses a failure of these conditions.  $\square$

**Lemma 5.8.** *Let  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  be a subdirectly irreducible kite with countably infinite  $I$  and  $J$ . Then, one of the following three cases must obtain:*

- (1)  $\lambda$  and  $\rho$  are bijections.
- (2)  $\lambda$  is a bijection and  $|I \setminus \rho(J)| = 1$ .
- (3)  $\rho$  is a bijection and  $|I \setminus \lambda(J)| = 1$ .

*Proof.* From the conditions in Theorem 5.5(2) it follows that  $\rho(J) \cup \lambda(J) = I$ . Suppose  $i, j \in \lambda(J) \setminus \rho(J)$ . Observe that any path of alternating  $\lambda$  and  $\rho$  between  $i$  and  $j$  must begin with  $\lambda^{-1}$ . Therefore, it must end with  $\rho$ , and so  $i = j$  and  $m = 0$ . So,  $|\lambda(J) \setminus \rho(J)| \leq 1$ . Similarly,  $|\rho(J) \setminus \lambda(J)| \leq 1$ . We have three cases to consider.

*Case 1.* Suppose  $|\lambda(J) \setminus \rho(J)| = 1$  and  $|\rho(J) \setminus \lambda(J)| = 1$ . Put  $i = \lambda(J) \setminus \rho(J)$  and  $j = \rho(J) \setminus \lambda(J)$ . Since  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is subdirectly irreducible, there is an  $m \in \omega$  such that  $(\rho \circ \lambda^{-1})^m(i) = j$ . For  $0 \leq n \leq m$ , put  $k_n = (\rho \circ \lambda^{-1})^n(i)$ . Take  $k \in J \setminus \{k_0, \dots, k_m\}$  and  $k_n$  with  $0 \leq n \leq m$ . It is not difficult to show, by case analysis, that no path of alternating  $\lambda$  and  $\rho$  can exist between  $k$  and  $k_n$ , unless  $i = j$ . But then either  $|\lambda(J) \setminus \rho(J)| \neq 1$  or  $|\rho(J) \setminus \lambda(J)| \neq 1$ , contradicting assumptions. This case is thus excluded.

*Case 2.* Suppose  $|\lambda(J) \setminus \rho(J)| = 0$  and  $|\rho(J) \setminus \lambda(J)| = 1$ . Then,  $\lambda(J) \subset \rho(J)$  and so  $I = \lambda(J) \cup \rho(J) = \rho(J)$ . Thus,  $\rho$  is a bijection and  $|I \setminus \lambda(J)| = 1$ , i.e., (3) holds. By symmetry, if  $|\rho(J) \setminus \lambda(J)| = 0$  and  $|\lambda(J) \setminus \rho(J)| = 1$ , we get that (2) holds.

*Case 3.* Finally, suppose  $|\lambda(J) \setminus \rho(J)| = 0 = |\rho(J) \setminus \lambda(J)|$ . Then  $\lambda(J) = \rho(J) = I$  and (1) holds.  $\square$

**5.1. Three infinite-dimensional kites.** It is consistent with Lemma 5.8 that no infinite-dimensional subdirectly irreducible kites exist. The examples below show that cases (1), (2) and (3) in Lemma 5.8 are non-void.

Case (1). Take  $I = J = \mathbb{Z}$  and put  $\lambda(i) = i$ ,  $\rho(i) = i + 1$ . The kite  $K_{\mathbb{Z}, \mathbb{Z}}^{\lambda, \rho}(\mathbb{Z})$  is subdirectly irreducible; its smallest nontrivial normal filter is the set of all sequences  $\langle k_i : i \in \mathbb{Z} \rangle$  with  $k_i \neq 0$  for finitely many  $i$ .

Case (2). Take  $I = J = \omega$  and put  $\lambda(i) = i$ ,  $\rho(i) = i + 1$ . The kite  $K_{\mathbb{Z}, \mathbb{Z}}^{\lambda, \rho}(\mathbb{Z})$  is subdirectly irreducible; its smallest nontrivial normal filter is the set of all sequences  $\langle k_i : i \in \omega \rangle$  with  $k_i \neq 0$  for finitely many  $i$ .

Case (3). Take  $I = J = \omega$  and put  $\lambda(i) = i + 1$ ,  $\rho(i) = i$ . We obtain an example which is symmetric, but not isomorphic, to the previous one.

**5.2. Classification.** Gathering all results of this section together, we obtain the classification of subdirectly irreducible kites promised at the beginning.

**Theorem 5.9.** *Let  $K_{I,J}^{\lambda, \rho}(\mathbf{G})$  be a subdirectly irreducible kite. Then,  $K_{I,J}^{\lambda, \rho}(\mathbf{G})$  is isomorphic to precisely one of:*

- (1)  $K_{n,n}^{\lambda, \rho}(\mathbf{G})$ , with  $\lambda(j) = j$  and  $\rho(j) = j + 1 \pmod{n}$ .
- (2)  $K_{\mathbb{Z}, \mathbb{Z}}^{\lambda, \rho}(\mathbf{G})$ , with  $\lambda(j) = j$  and  $\rho(j) = j + 1$ .
- (3)  $K_{\omega, \omega}^{\lambda, \rho}(\mathbf{G})$ , with  $\lambda(j) = j$  and  $\rho(j) = j + 1$ .
- (4)  $K_{\omega, \omega}^{\lambda, \rho}(\mathbf{G})$ , with  $\lambda(j) = j + 1$  and  $\rho(j) = j$ .
- (5)  $K_{n+1, n}^{\lambda, \rho}(\mathbf{G})$ , with  $\lambda(j) = j$  and  $\rho(j) = j + 1$ .

Moreover, types (1) and (2) consist entirely of pseudo MV-algebras, the other types contain no pseudo MV-algebras except the two-element Boolean algebra. A kite of type (3) or (4) is good if and only if it is a two-element Boolean algebra. A kite of type (5) is good if and only if  $J = \emptyset$ .

*Proof.* The cases with  $I$  and  $J$  finite follow from Lemma 5.6. The infinite-dimensional cases can be derived from Lemma 5.8 using arguments mimicking those of Lemma 5.6. The ‘moreover’ statements follow from Lemma 3.4.  $\square$

For kites of the types from Theorem 5.9 above it will be convenient from now on to use the following notational convention:

- (1)  $K_{n,n}^{0,1}(\mathbf{G})$
- (2)  $K_{\mathbb{Z}, \mathbb{Z}}^{0,1}(\mathbf{G})$
- (3)  $K_{\omega, \omega}^{0,1}(\mathbf{G})$
- (4)  $K_{\omega, \omega}^{1,0}(\mathbf{G})$
- (5)  $K_{n+1, n}^{0,1}(\mathbf{G})$

where the upper indices correspond to the functions  $\lambda$  and  $\rho$  in such a way that whenever  $\lambda$  or  $\rho$  is the identity function, we replace it by 0; otherwise we replace it by 1 (which sits well with the fact that it is then a kind of successor function).

**Corollary 5.10.** *A subdirectly irreducible kite is good if and only if it is either a pseudo MV-algebra or it is of the form  $K_{1,0}^{0,0}(\mathbf{G})$ , for a subdirectly irreducible  $\ell$ -group  $\mathbf{G}$ .*

The last result in this section shows that subdirectly irreducible kites are building blocks from which all kites can be built. Namely, we will show that each kite is a subdirect product of subdirectly irreducible ones. This is not a trivial corollary of Birkhoff’s subdirect representation theorem, the value added is that the subdirect factors are kites as well.

**Theorem 5.11.** *Let  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  be a kite. Then,  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is subdirectly embeddable into a product of kites of the form  $K_{I',J'}^{\lambda,\rho}(\mathbf{G})$ , where  $I' \cup J'$  is a connected component of the graph of  $\lambda \circ \rho^{-1} \cup \rho \circ \lambda^{-1}$  paths in  $I \cup J$ .*

*Proof.* It is not difficult to show, using Lemmas 5.2 and 5.3, that if the  $\ell$ -group  $\mathbf{G}$  is not subdirectly irreducible, then any subdirect representation  $\mathbf{G} \leq \prod_{s \in S} \mathbf{G}_s$  naturally gives rise to a subdirect representation  $K_{I,J}^{\lambda,\rho}(\mathbf{G}) \leq \prod_{s \in S} K_{I,J}^{\lambda,\rho}(\mathbf{G}_s)$ . We leave the details to the reader.

Now, assume  $\mathbf{G}$  is subdirectly irreducible and let  $P$  stand for the graph of  $\lambda \circ \rho^{-1} \cup \rho \circ \lambda^{-1}$  paths in  $I \cup J$ . Let further  $\mathcal{C}$  be the set of all connected components of  $P$ . From Theorem 5.5 it follows that for each connected component  $C = I_C \cup J_C$  of  $P$  the kite  $K_{I_C,J_C}^{\lambda,\rho}(\mathbf{G})$  is subdirectly irreducible. For each  $C \in \mathcal{C}$  let  $N_C$  be the set of all  $a = \langle a_i^{-1} \in I \rangle \in (G^-)^I$  such that  $i \notin C$  implies  $a_i = e$ . It is straightforward to see that  $N_C$  is a normal filter for each  $C \in \mathcal{C}$ , and that  $K_{I,J}^{\lambda,\rho}(\mathbf{G})/\Theta_C$ , where  $\Theta_C$  is the congruence corresponding to  $N_C$ , is isomorphic to  $K_{I_C,J_C}^{\lambda,\rho}(\mathbf{G})$ . Since  $\bigcap_{C \in \mathcal{C}} N_C = \{1\}$ , this proves claim.  $\square$

**Corollary 5.12.** *Let  $\mathbf{K}$  be the variety generated by all kites. Then,  $\mathbf{K}$  is generated by all subdirectly irreducible kites.*

## 6. FINITE-DIMENSIONAL KITES

A kite  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  will be called *n-dimensional* if  $|I| = n \in \omega$ , and *finite-dimensional* if it is *n-dimensional* for some  $n$ . We write  $\mathcal{K}_n$  for the class of all *n-dimensional* kites and  $\mathbf{K}_n$  for the variety generated by  $\mathcal{K}_n$ . In this section we will show that  $\mathbf{K}$  is generated by finite-dimensional kites, and thus  $\mathbf{K}$  is the varietal join of  $\mathbf{K}_n$  for  $n \in \omega$ . Our proof proceeds by embedding infinitely dimensional kites from Theorem 5.9 into a quotient of a subalgebra of a product of kites of finitely dimensional kites.

For any  $n \in \omega$ , take the kite  $K_{2n+1,2n+1}^{0,1}(\mathbf{G})$ . We think of the set  $2n+1$  as the universe of the additive group  $\mathbb{Z}/(2n+1)\mathbb{Z}$  and label its elements accordingly so that  $2n+1 = \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}$ . Consider the direct product  $\prod_{n \in \omega} K_{2n+1,2n+1}^{0,1}(\mathbf{G})$ . As usual, for  $u \in \prod_{n \in \omega} K_{2n+1,2n+1}^{0,1}(\mathbf{G})$ , we will write  $u(i)$  for the  $i$ -th element of  $u$ . Then, employing our numbering convention, we will write  $u(i)$  as  $\langle u_{-i}, \dots, u_0, \dots, u_i \rangle$ , and so we have

$$u = \langle \langle u_{-n}, \dots, u_0, \dots, u_n \rangle : n \in \omega \rangle.$$

It is easy to see that the set  $S_{\mathbf{G}} = U_{\mathbf{G}} \cup L_{\mathbf{G}}$ , where

$$\begin{aligned} U_{\mathbf{G}} &= \{ \langle a_{-n}^{-1}, \dots, a_i^{-1}, \dots, a_n^{-1} \rangle : a_i^{-1} \in G^-, n \in \omega \} \\ L_{\mathbf{G}} &= \{ \langle f_{-n}, \dots, f_i, \dots, f_n \rangle : f_i \in G^+, n \in \omega \} \end{aligned}$$

is a subuniverse of  $\prod_{n \in \omega} K_{2n+1,2n+1}^{0,1}(\mathbf{G})$ . Let  $S_{\mathbf{G}}$  be the corresponding subalgebra of  $\prod_{n \in \omega} K_{2n+1,2n+1}^{0,1}(\mathbf{G})$ . Now, on  $S_{\mathbf{G}}$  we define a binary relation, putting

$$u \sim w \quad \text{iff} \quad \exists k \in \omega \forall n \geq k : \llbracket u(n) \neq w(n) \rrbracket \cap [-n+k, n-k] = \emptyset$$

where  $\llbracket u(n) \neq w(n) \rrbracket = \{-n \leq i \leq n : u_i \neq w_i\}$ , as usual. Intuitively,  $u \sim w$  holds if, for sufficiently large  $n \in \omega$ , the sequences  $u(n)$  and  $w(n)$  differ only at a bounded number of initial and final places. It should be clear that  $u \sim w$  can hold only if either  $u, w \in U_{\mathbf{G}}$  or  $u, w \in L_{\mathbf{G}}$ .

**Lemma 6.1.** *The relation  $\sim$  is a congruence on  $\prod_{n \in \omega} K_{2n+1, 2n+1}^{0,1}(\mathbf{G})$ .*

*Proof.* Reflexivity and symmetry are obvious. For transitivity, suppose  $u \sim v$  and  $v \sim w$ . By definition then, there are  $k_1, k_2 \in \omega$  such that

$$\forall n \geq k_1: \llbracket u(n) \neq v(n) \rrbracket \cap [-n + k_1, n - k_1] = \emptyset$$

and

$$\forall n \geq k_2: \llbracket v(n) \neq w(n) \rrbracket \cap [-n + k_2, n - k_2] = \emptyset.$$

Then, putting  $k = \max\{k_1, k_2\}$  we obtain

$$\forall n \geq k: \llbracket u(n) \neq w(n) \rrbracket \cap [-n + k, n - k] = \emptyset$$

and so  $u \sim w$  as required. It remains to show that  $\sim$  preserves the operations. We will only prove it for two cases of multiplication. Let  $u \sim w$  and  $v \sim s$ , with  $u, w \in L_{\mathbf{G}}$  and  $v, s \in U_{\mathbf{G}}$ . Thus,

$$\begin{aligned} u(n) &= \langle f_{-n}, \dots, f_i, \dots, f_n \rangle \\ w(n) &= \langle g_{-n}, \dots, g_i, \dots, g_n \rangle \\ v(n) &= \langle a_{-n}^{-1}, \dots, a_i^{-1}, \dots, a_n^{-1} \rangle \\ s(n) &= \langle b_{-n}^{-1}, \dots, b_i^{-1}, \dots, b_n^{-1} \rangle \end{aligned}$$

with  $a_i^{-1}, b_i^{-1} \in G^-$  and  $f_i, g_i \in G^+$ , for any  $n \in \omega$ . Further, by definition of  $\sim$ , there are  $k_1, k_2 \in \omega$  such that

$$\begin{aligned} \forall n \geq k_1: \llbracket u(n) \neq w(n) \rrbracket \cap [-n + k_1, n - k_1] &= \emptyset \\ \forall n \geq k_2: \llbracket v(n) \neq s(n) \rrbracket \cap [-n + k_2, n - k_2] &= \emptyset. \end{aligned}$$

Consider  $uv$  and  $ws$ .

$$\begin{aligned} uv &= \langle u(n): n \in \omega \rangle \cdot \langle v(n): n \in \omega \rangle \\ &= \langle u(n) \cdot v(n): n \in \omega \rangle \\ &= \langle \langle f_{-n}, \dots, f_n \rangle \cdot \langle a_{-n}^{-1}, \dots, a_n^{-1} \rangle: n \in \omega \rangle \\ &= \langle \langle f_{-n} a_{-n+1}^{-1} \vee e, \dots, f_{n-1} a_n^{-1} \vee e, f_n a_{n+1}^{-1} \vee e \rangle: n \in \omega \rangle \\ &= \langle \langle f_{-n} a_{-n+1}^{-1} \vee e, \dots, f_{n-1} a_n^{-1} \vee e, f_n a_{-n}^{-1} \vee e \rangle: n \in \omega \rangle \end{aligned}$$

and similarly,

$$\begin{aligned} ws &= \langle w(n) \cdot s(n): n \in \omega \rangle \\ &= \langle \langle g_{-n} b_{-n+1}^{-1} \vee e, \dots, g_{n-1} b_n^{-1} \vee e, g_n b_{-n}^{-1} \vee e \rangle: n \in \omega \rangle. \end{aligned}$$

Now, for a given  $n \geq k = \max\{k_1, k_2\}$ , let us compare

$$\begin{aligned} u(n) \cdot v(n) &= \langle f_{-n} a_{-n+1}^{-1} \vee e, \dots, f_n a_{-n}^{-1} \vee e \rangle \\ w(n) \cdot s(n) &= \langle g_{-n} b_{-n+1}^{-1} \vee e, \dots, g_n b_{-n}^{-1} \vee e \rangle. \end{aligned}$$

From the definition of multiplication on  $K_{2n+1, 2n+1}^{0,1}(\mathbf{G})$  it follows immediately that the two can differ only at initial places up to  $f_{-n+k} a_{-n+k+1}^{-1}$ ,  $g_{-n+k} b_{-n+k+1}^{-1}$ , respectively, and at final places from  $f_{n-k-1} a_{-n-k}^{-1}$ ,  $g_{n-k-1} b_{-n-k}^{-1}$ , respectively. It follows that

$$\forall n \geq k+1: \llbracket u(n) \cdot v(n) \neq w(n) \cdot s(n) \rrbracket \cap [-n + k + 1, n - k - 1] = \emptyset$$



and thus  $uv \sim ws$  as required. Other cases of multiplication are similar, and analogous arguments prove that  $\sim$  preserves all the operations, completing the proof.  $\square$

We are now ready to show that  $K_{\mathbb{Z},\mathbb{Z}}^{0,1}(\mathbf{G})$  embeds into  $\mathbf{S}_{\mathbf{G}}/\sim$ . To this end, first define a map  $\mu^-: K_{\mathbb{Z},\mathbb{Z}}^{0,1}(\mathbf{G}) \rightarrow \mathbf{S}_{\mathbf{G}}$  putting

$$\langle u_i: i \in \mathbb{Z} \rangle \mapsto \langle \langle u_{-n}, \dots, u_0, \dots, u_n \rangle, n \in \omega \rangle$$

where  $\langle u_i: i \in \mathbb{Z} \rangle$  is either  $\langle a_i^{-1}: i \in \mathbb{Z} \rangle$  with  $a_i \in G^-$  for each  $i$ , or  $\langle f_i: i \in \mathbb{Z} \rangle$  with  $f_i \in G^+$  for each  $i$ . The map  $\mu^-$  extends naturally to a map  $\mu: K_{\mathbb{Z},\mathbb{Z}}^{0,1}(\mathbf{G}) \rightarrow \mathbf{S}_{\mathbf{G}}/\sim$ , namely  $\mu = \mu^-/\sim$ .

**Lemma 6.2.** *The map  $\mu$  above is an embedding of  $K_{\mathbb{Z},\mathbb{Z}}^{0,1}(\mathbf{G})$  into  $\mathbf{S}_{\mathbf{G}}/\sim$ .*

*Proof.* To show that  $\mu$  is one-one, let  $u = \langle u_i: i \in \mathbb{Z} \rangle$  and  $w = \langle w_i: i \in \mathbb{Z} \rangle$  be such that  $u_i = w_i$  for all  $i \neq \ell$  and  $u_\ell \neq w_\ell$ , where  $\ell \in \mathbb{Z}$  is arbitrary, but fixed. By definition,  $\mu(u) \neq \mu(w)$  iff  $\mu^-(u) \not\sim \mu^-(w)$ . Now, observe that for any  $k \in \omega$ , if we take  $n \geq k + |\ell| + 1$ , then  $-n + k < \ell < n - k$ , and therefore  $\ell \in \llbracket u(n) \neq w(n) \rrbracket$ . It follows that

$$\forall k \exists n \geq k: \llbracket u(n) \neq w(n) \rrbracket \cap [-n + k, n - k] \neq \emptyset$$

and so  $\mu^-(u) \not\sim \mu^-(w)$  as needed.

To prove that  $\mu$  is a homomorphism we proceed case by case. We only present two cases of multiplication here. Let  $u = \langle a_i^{-1}: i \in \mathbb{Z} \rangle$  and  $w = \langle f_i: i \in \mathbb{Z} \rangle$ . Then,  $uw = \langle a_i^{-1}f_i \vee e: i \in \mathbb{Z} \rangle$  and  $wu = \langle f_ia_{i+1}^{-1} \vee e: i \in \mathbb{Z} \rangle$ . Thus,

$$\mu^-(uw) = \langle \langle a_{-n}^{-1}f_{-n} \vee e, \dots, a_0^{-1}f_0 \vee e, \dots, a_n^{-1}f_n \vee e \rangle, n \in \omega \rangle$$

and

$$\mu^-(wu) = \langle \langle f_{-n}a_{-n+1}^{-1} \vee e, \dots, f_0a_1^{-1} \vee e, \dots, f_na_{n+1}^{-1} \vee e \rangle, n \in \omega \rangle$$

On the other hand, we have

$$\begin{aligned} \mu^-(u)\mu^-(w) &= \langle \langle a_{-n}^{-1}, \dots, a_0^{-1}, \dots, a_n^{-1} \rangle, n \in \omega \rangle \cdot \langle \langle f_{-n}, \dots, f_0, \dots, f_n \rangle, n \in \omega \rangle \\ &= \langle \langle a_{-n}^{-1}f_{-n} \vee e, \dots, a_0^{-1}f_0 \vee e, \dots, a_n^{-1}f_n \vee e \rangle, n \in \omega \rangle \end{aligned}$$

and also,

$$\begin{aligned} \mu^-(w)\mu^-(u) &= \langle \langle f_{-n}, \dots, f_0, \dots, f_n \rangle, n \in \omega \rangle \cdot \langle \langle a_{-n}^{-1}, \dots, a_0^{-1}, \dots, a_n^{-1} \rangle, n \in \omega \rangle \\ &= \langle \langle f_{-n}a_{-n+1}^{-1} \vee e, \dots, f_0a_1^{-1} \vee e, \dots, f_na_{n+1}^{-1} \vee e \rangle, n \in \omega \rangle \\ &= \langle \langle f_{-n}a_{-n+1}^{-1} \vee e, \dots, f_0a_1^{-1} \vee e, \dots, f_na_{-n}^{-1} \vee e \rangle, n \in \omega \rangle. \end{aligned}$$

Thus, for every  $n \in \omega$  we have that  $(\mu^-(uw))(n) = (\mu^-(u)\mu^-(w))(n)$  and the sequences  $(\mu^-(wu))(n)$  and  $(\mu^-(w)\mu^-(u))(n)$  differ at most at the final place. It follows that

$$\llbracket (\mu^-(uw))(n) \neq (\mu^-(u)\mu^-(w))(n) \rrbracket \cap [-n + 1, n - 1] = \emptyset$$

and

$$\llbracket (\mu^-(wu))(n) \neq (\mu^-(w)\mu^-(u))(n) \rrbracket \cap [-n + 1, n - 1] = \emptyset$$

proving  $\mu^-(uw) \sim \mu^-(u)\mu^-(w)$  and  $\mu^-(wu) \sim \mu^-(w)\mu^-(u)$ , as required.  $\square$

Next we turn to  $K_{\omega,\omega}^{0,1}(\mathbf{G})$ . The argument is by and large analogous to the one just used, but simpler. Take the direct product  $\prod_{n \in \omega} K_{n+1,n}^{0,1}(\mathbf{G})$ . We have that  $S'_{\mathbf{G}} = U'_{\mathbf{G}} \cup L'_{\mathbf{G}}$ , where

$$\begin{aligned} U'_{\mathbf{G}} &= \{ \langle a_0^{-1}, \dots, a_i^{-1}, \dots, a_{n+1}^{-1} \rangle, : a_i^{-1} \in G^-, n \in \omega \} \\ L'_{\mathbf{G}} &= \{ \langle f_0, \dots, f_i, \dots, f_n \rangle : f_i \in G^+, n \in \omega \} \end{aligned}$$

is a subuniverse of  $\prod_{n \in \omega} K_{n+1,n}^{0,1}(\mathbf{G})$ . Let  $S'_{\mathbf{G}}$  be the corresponding subalgebra of  $\prod_{n \in \omega} K_{n+1,n}^{0,1}(\mathbf{G})$ . Here we can embed  $K_{\omega,\omega}^{0,1}(\mathbf{G})$  directly into  $S'_{\mathbf{G}}$ . Define a map  $\nu: K_{\omega,\omega}^{0,1}(\mathbf{G}) \rightarrow S'_{\mathbf{G}}$  putting

$$\langle f_i : i \in \omega \rangle \mapsto \langle \langle f_0, \dots, f_n \rangle, n \in \omega \rangle$$

where  $f_i \in G^+$  for all  $i$ , and

$$\langle a_i^{-1} : i \in \omega \rangle \mapsto \langle \langle a_0^{-1}, a_1^{-1}, \dots, a_{n+1}^{-1} \rangle, n \in \omega \rangle$$

where  $a_i^{-1} \in G^-$  for all  $i$ . It is then rather straightforward to prove the following lemma.

**Lemma 6.3.** *The map  $\nu$  above is an embedding of  $K_{\omega,\omega}^{0,1}(\mathbf{G})$  into  $S'_{\mathbf{G}}$ .*

Finally, we deal with  $K_{\omega,\omega}^{1,0}(\mathbf{G})$ . We again use the algebra  $S''_{\mathbf{G}}$ , but this time define a map  $\nu': K_{\omega,\omega}^{1,0}(\mathbf{G}) \rightarrow S'_{\mathbf{G}}$  by reversing the ordering in  $\nu$ , namely, by

$$\langle f_i : i \in \omega \rangle \mapsto \langle \langle f_n, f_{n-1}, \dots, f_0 \rangle, n \in \omega \rangle$$

where  $f_i \in G^+$  for all  $i$ , and

$$\langle a_i^{-1} : i \in \omega \rangle \mapsto \langle \langle a_{n+1}^{-1}, a_n^{-1}, \dots, a_0^{-1} \rangle, n \in \omega \rangle$$

where  $a_i^{-1} \in G^-$  for all  $i$ . We obtain the lemma below.

**Lemma 6.4.** *The map  $\nu'$  above is an embedding of  $K_{\omega,\omega}^{1,0}(\mathbf{G})$  into  $S'_{\mathbf{G}}$ .*

**Theorem 6.5.** *The variety  $\mathbf{K}$  is generated by all finite-dimensional kites.*

*Proof.* By Corollary 5.12, the variety  $\mathbf{K}$  is generated by all subdirectly irreducible kites. Let  $\mathcal{K}_{\text{fin}}$  be the class of all finitely dimensional kites. By Lemmas 6.2, 6.3, and 6.4, every subdirectly irreducible kite belongs to  $SHSP(\mathcal{K}_{\text{fin}})$ . Thus  $\mathbf{K} = V(\mathcal{K}_{\text{fin}})$  as claimed.  $\square$

**Corollary 6.6.** *The variety  $\mathbf{K}$  generated by all kites is the varietal join of varieties  $\mathbf{K}_n$ , generated by  $n$ -dimensional kites. Briefly,*

$$\mathbf{K} = \bigvee_{n \in \omega} \mathbf{K}_n.$$

## 7. APPLICATIONS

Let  $\mathbf{A} = (A; \backslash, /, \cdot, 0, 1)$  be a pseudo BL-algebra and let  $g \in A \setminus \{1\}$ . Then there is a filter,  $V$ , of  $A$  not containing  $g$  and is maximal with respect to this property. We call it a *value* of  $g$ , and the filter  $V^*$  generated by  $V$  and the element  $g$  is said to be a *cover* of  $V$ . As in  $\ell$ -groups, we say that  $\mathbf{A}$  is *normal-valued* if every value is normal in its cover. Let  $\mathbf{NVpsBL}$  be the class of normal-valued pseudo BL-algebras. In [DGK] it was proved that  $\mathbf{NVpsBL}$  is a variety, and in contrast to the variety of  $\ell$ -groups, it is not the greatest proper subvariety of the variety of  $\ell$ -groups because

NVpsBL is a proper subvariety of the system of all pseudo BL-algebras  $\mathbf{A}$  such that every maximal filter of  $\mathbf{A}$  is normal that is also a variety, see [DGK].

According to Wolfenstein [Dar, Thm 41.1], an  $\ell$ -group  $G$  is normal-valued iff every  $a, b \in G^-$  satisfy  $a^2 b^2 \leq ba$ , or in our language

$$a^2 \cdot b^2 \leq b \cdot a. \quad (1)$$

In [BDK] it was shown a large variety of pseudo BL-algebras that are normal-valued iff they satisfy (1). We note that the same is true for kites.

**Theorem 7.1.** *Let  $\mathbf{G}$  be an  $\ell$ -group. Then the kite  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$  is normal-valued if and only if (1) holds for all  $a, b \in K_{I,J}^{\lambda,\rho}(\mathbf{G})$ .*

*Proof.* It follows from the definition of kites and Wolfenstein's criterion, [Dar, Thm 41.1].  $\square$

We note that if  $x \in K_{I,J}^{\lambda,\rho}(\mathbf{G})$ , then

$$x^2 = 0 \quad \text{or} \quad (\sim x)^2 = 0. \quad (2)$$

Let  $V(K_{I,J}^{\lambda,\rho}(\mathbf{G}))$  denote the variety of pseudo BL-algebras generated by a kite  $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ .

Let  $n \geq 2$  be a fixed integer and we set  $J_n = \{1, \dots, n\}$ ,  $I_n = \{1, \dots, n, n+1\}$ ,  $J'_n = \{2, \dots, n\}$ , and  $I'_n = \{2, \dots, n, n+1\}$ . We set  $\lambda, \rho : J_n \rightarrow I_n$  by  $\lambda(i) = i$  and  $\rho(i) = i+1$  for each  $i \in J_n$ . Take two terms  $f_{\sim}(x) := \sim x$  and  $f_{-}(x) = -x$ . Then on  $K_{I_n, J_n}^{\lambda_n, \rho_n}(\mathbf{G})$  we have

$$f_{\sim}^{2n+1}(x) \in \{0, 1\} = B(K_{I_n, J_n}^{\lambda_n, \rho_n}(\mathbf{G})) \quad (3)$$

for every  $x \in K_{I_n, J_n}^{\lambda_n, \rho_n}(\mathbf{G})$ .

For abbreviation, given an integer  $n \geq 1$ , we set  $\mathbb{Z}_n^{\dagger} = K_{I_n, J_n}^{\lambda_n, \rho_n}(\mathbf{Z})$ . In addition, we define  $\mathbb{Z}_0^{\dagger} := K_{I_1, J_1}^{\lambda_1, \rho_1}(\mathbf{O})$ , where  $\mathbf{O}$  is the trivial  $\ell$ -group consisting only from the identity. Then  $\mathbb{Z}_0^{\dagger}$  is the two-element Boolean algebra, therefore,  $\mathbb{Z}_0^{\dagger}$  generates the variety of Boolean algebras, BA.

**Theorem 7.2.** *For any integer  $n \geq 1$ ,  $V(\mathbb{Z}_n^{\dagger})$  is a cover of the variety BA, and  $n \neq m$  implies  $V(\mathbb{Z}_n^{\dagger}) \neq V(\mathbb{Z}_m^{\dagger})$ .*

*Proof.* For  $n = 0$ , the variety  $V(\mathbb{Z}_n^{\dagger})$  is precisely the variety of product logic algebras and it is well-known that this variety covers BA. For  $n = 1$ , it was proved in [JiMo, Thm 11] that the variety  $V(\mathbb{Z}_1^{\dagger})$  covers BA. Thus, we can assume that  $n \geq 1$ .

It follows from Theorem 5.9 that  $\mathbb{Z}_n^{\dagger}$  is subdirectly irreducible.

*Claim 1.* *Any nontrivial element of  $\mathbb{Z}_n^{\dagger}$  generates a subalgebra of  $\mathbb{Z}_n^{\dagger}$  that is an isomorphic copy of  $\mathbb{Z}_n^{\dagger}$ .*

Assume  $a = \langle a_1^{-1}, \dots, a_{n+1}^{-1} \rangle$  is an element from  $(G^-)^I$  and let  $A$  be the subalgebra of  $\mathbb{Z}_n^{\dagger}$  generated by  $a$ . Let  $i_0$  be the first index such that is different of  $e^{-1}$ . There is an integer  $n_0$  such that  $f_{\sim}^{n_0}(a) = \langle e^{-1}, \dots, e^{-1}, a_{i_0}^{-1} \rangle \in A$ . There is also an integer  $m$  such that  $f_{-}^m(\langle e^{-1}, \dots, e^{-1}, a_{i_0}^{-1} \rangle) = \langle a_{i_0}, e^{-1}, \dots, e^{-1} \rangle \in A$ . In addition, for every  $i = 1, \dots, n+1$ , the element  $x_i = \langle b_1^{-1}, \dots, b_{n+1}^{-1} \rangle$  belongs to  $A$ , where  $b_i^{-1} = a_{i_0}^{-1}$  and  $b_j^{-1} = e^{-1}$  for  $j \neq i$ .

Since  $\sim\langle a_1^{-1}, \dots, a_{n+1}^{-1} \rangle = \sim\langle a_1^{-1}, \dots, a_n^{-1} \rangle$ ,  $\langle f_1, \dots, f_n \rangle = \langle e^{-1}, f_1^{-1}, \dots, f_n^{-1} \rangle$ , we see that the algebra  $A$  can be generated equivalently by some appropriate element from  $(G^-)^I$  or from  $(G^+)^J$ .

Hence, if  $a = \text{g.c.d.}\{a_1, \dots, a_{n+1}\}$ , we see that  $A$  is generated e.g. by the element  $\langle a^{-1}, e^{-1}, \dots, e^{-1} \rangle$ , and it is an isomorphic copy of  $\mathbb{Z}_n^\dagger$ .

*Claim 2. If  $\mathbf{A}$  is an algebra from  $\mathbf{V}(\mathbb{Z}_n^\dagger)$ , then for any  $x \in A$ , we have*

$$f_{\sim}^{2n+1}(x) \in B(A). \quad (4)$$

Indeed, let  $C$  be a subdirectly irreducible algebra from  $\mathbf{V}(\mathbb{Z}_n^\dagger)$ . By the Jónsson Lemma,  $C$  is a homomorphic image of a subalgebra  $D$  of an ultrapower  $\mathcal{U}$  of  $\mathbb{Z}_n^\dagger$  on the index set  $U$ . By (3), if  $x \in \mathbb{Z}_n^\dagger$ , then  $f_{\sim}^{2n+1}(x) \in \{0, 1\} = B(\mathbb{Z}_n^\dagger)$ . If  $x \in \langle t_u : u \in U \rangle$ , then  $f_{\sim}^{2n+1}(x) \in B((\mathbb{Z}_n^\dagger)^U)$ . Similarly, if  $x \in \langle t_u : u \in U \rangle / \mathcal{U}$ ,  $f_{\sim}^{2n+1}(x) \in B((\mathbb{Z}_n^\dagger)^U / \mathcal{U})$ . In the same way, we can prove that if  $x \in C$ , then  $f_{\sim}^{2n+1}(x) \in B(C)$ . The general case follows from the statement follows from the fact that  $A$  is isomorphic to a subdirect product of algebras from  $HSP_U(\mathbb{Z}_n^\dagger)$ .

*Claim 3. Any subdirectly irreducible algebra of  $\mathbf{V}(\mathbb{Z}_n^\dagger)$  is either the two-element Boolean algebra or has a subalgebra isomorphic to  $\mathbb{Z}_n^\dagger$ .*

Let  $C$  be a subdirectly irreducible algebra of  $\mathbf{V}(\mathbb{Z}_n^\dagger)$  and assume that it has a nontrivial element  $0 < c < 1$ . Now we use again the Jónsson Lemma, and we assume that  $C$  is a homomorphic image of a subalgebra  $D$  of an ultrapower  $\mathcal{U}$  of  $\mathbb{Z}_n^\dagger$  on the index set  $U$ . Let  $d \in D$  be a preimage of  $c$  under the homomorphism. We show that  $a = d_{\sim}$  generates a subalgebra of  $D$  that is isomorphic to  $\mathbb{Z}_n^\dagger$ . Then  $a = \langle a_u : u \in U \rangle / \mathcal{U}$ . Consider the set  $V = \{u \in U : a_u^2 = 0\}$ . If  $V \in \mathcal{U}$ , then  $a$  generates in the same way as  $\langle 1, \dots, 1 \rangle \in (\mathbb{Z}^+)^n$ . If  $V \notin \mathcal{U}$ , we can use the generator like  $\langle 0, \dots, 0, -1 \rangle \in (\mathbb{Z}^-)^{n+1}$ .

*Claim 4. Every  $\mathbf{V}(\mathbb{Z}_n^\dagger)$  is a cover variety of the variety of Boolean algebras.*

Let  $\mathcal{V}$  be a subvariety of  $\mathbf{V}(\mathbb{Z}_n^\dagger)$  containing properly the variety of Boolean algebras, and let  $A$  be a subdirectly irreducible algebra from  $\mathcal{V}$ . By Claim 3,  $A$  is either a two-element Boolean algebra or contains an isomorphic copy of  $\mathbb{Z}_n^\dagger$ . Therefore, in the later case,  $A \notin \mathbf{BA}$  and we have  $\mathbf{V}(A) = \mathbf{V}(\mathbb{Z}_n^\dagger) \subseteq \mathcal{V}$ , which proves the statement of the Theorem.

*Claim 5. If  $n$  and  $m$  are two different positive integers, then  $\mathbf{V}(\mathbb{Z}_n^\dagger) \neq \mathbf{V}(\mathbb{Z}_m^\dagger)$ .*

Assume  $n < m$ . Verifying (4), we see that the element  $x = \langle -1, \dots, -1 \rangle$  from  $\mathbb{Z}_m^\dagger$  does not belong to  $\mathbf{V}(\mathbb{Z}_n^\dagger)$ .  $\square$

The next result follows immediately from Lemma 3.4, Theorem 7.2, and the subdirect representation theorem.

**Corollary 7.3.** *An algebra  $\mathbf{A} \in \mathbf{V}(\mathbb{Z}_n^\dagger)$ , for any  $n \in \omega$ , is good if and only if  $\mathbf{A}$  is a Boolean algebra.*

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